Trapping regions and an ODE-type proof of the existence and uniqueness theorem for Navier-Stokes equations with periodic boundary conditions on the plane

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Abstract

Using ODE-methods and trapping regions derived by Mattingly and Sinai we give a new proof of the existence and uniqueness of solutions to Navier-Stokes equations with periodic boundary conditions on the plane.

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1 Introduction.

The goal of this paper to present self-contained account of the ODE-type proofs from [ES, MS, S] of the existence and uniqueness of the Navier-Stokes systems with periodic boundary conditions on the plane. Mattingly and Sinai called their proof elementary (see title of [MS]), but their proof was ODE-type (elementary in their sense) only up to the moment of getting the trapping regions for all Galerkin projections, but to pass to the limit with the dimensions of Galerkin projections they invoked the now standard results from [CF, DG, T] (which are not elementary in any sense), which are usually not mastered by the researchers working in dynamics of ODE's, to which this note is mainly addressed. Here we

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fill-in this gap by giving ODE-type arguments, which allow to pass to the limit. Using ODE-type estimates based on the logarithmic norms we also obtained uniqueness and an estimate for the Lipschitz constant of evolution induced by the Navier-Stokes equations . In fact we have proved that on the trapping region we have semidynamical system. The results we prove here are well known for Navier-Stokes system in 2D (see for example [FT, ES, K, DT]), but the method of getting estimates on Galerkin projections presented in section 5 appears to be new.

Another goal of this paper is to prepare the ground for the rigorous study of the dynamics of the Navier-Stokes equations with periodic boundary conditions. The trapping regions described here are particular examples of the self-consistent apriori bounds introduced in [ZM] for the rigorous study of the dynamics of the dissipative PDE's.

A few words about a general construction of the paper: In sections 2 and 3 we recall the results from [ES, MS, S] about the trapping regions. Sections 4 and 5 contain ODE-type proofs of the convergence of the Galerkin scheme on trapping regions. The remaining sections contain the existence results for the Navier-Stokes equations on the plane and the Sannikov and Kaloshin [S] result in the dimension three.

2 Navier-Stokes equations

The general d-dimensional Navier-Stokes system (NSS) is written for d unknown functions $u(t,x) = (u_1(t,x),\dots,u_d(t,x))$ of d variables $x = (x_1,\dots,x_d)$ and time t and the pressure $p(t,x)$.

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{d} u_k \frac{\partial u_i}{\partial x_k} = \nu \triangle u_i - \frac{\partial p}{\partial x_i} + f^{(i)} \tag{1}$$

$$\text{div } u = \sum_{i=1}^{d} \frac{\partial u_i}{\partial x_i} = 0 \tag{2}$$

The functions $f^{(i)}$ are the components of the external forcing, $\nu > 0$ is the viscosity.

We consider (1),(2) on the torus $\mathbb{T}^d = (\mathbb{R}/2\pi)^d$. This allows us to use Fourier series. We write

$$u(t,x) = \sum_{k\in\mathbb{Z}^d} u_k(t) e^{i(k,x)}, \qquad p(t,x) = \sum_{k\in\mathbb{Z}^d} p_k(t) e^{i(k,x)} \tag{3}$$

Observe that $u_k(t) \in \mathbb{R}^d$, i.e. they are d-dimensional vectors and $p_k(t) \in \mathbb{R}$. We will always assume that $f_0 = 0$ and $u_0 = 0$.

Observe that (2) is reduced to the requirement $u_k \perp k$. Namely

$$\text{div } u = \sum_{k\in\mathbb{Z}^d} i(k, u_k(t)) e^{i(k,x)} = 0$$

$$(k, u_k) = 0 \quad k \in \mathbb{Z}^d$$

To derive the evolution equation for $u_k(t)$ we will compute now the nonlinear term in (1). We will use the following notation $u_k = (u_{k,1}, \ldots, u_{k,d})$

$$\sum_l u_l \frac{\partial u}{\partial x_l} = \left(\sum_{k_1,l} u_{k_1,l} e^{i(k_1,x)} \right) \left(\sum_{k_2} i k_{2,l} u_{k_2} e^{i(k_2,x)} \right) = \tag{4}$$

$$= i \sum_{l,k_1,k_2} e^{i(k_1+k_2,x)} k_{2l} \cdot u_{k_1,l} \cdot u_{k_2} = i \sum_{k_1,k_2} e^{i(k_1+k_2,x)} (k_2|u_{k_1}) u_{k_2} = \tag{5}$$

$$i \sum_{k \in \mathbb{Z}^d} \left(\sum_{k_1} (u_{k_1}|k - k_1) u_{k-k_1} \right) e^{i(k,x)} = i \sum_{k \in \mathbb{Z}^d} \left(\sum_{k_1} (u_{k_1}|k) u_{k-k_1} \right) e^{i(k,x)} \tag{6}$$

We obtain the following infinite ladder of differential equations for u_k

$$\frac{du_k}{dt} = -i \sum_{k_1} (u_{k_1}|k) u_{k-k_1} - \nu k^2 u_k - i p_k k + f_k \tag{7}$$

Here f_k are components of the external forcing. Let $\sqcap_k$ denote the operator of orthogonal projection to the $(d-1)$-dimensional plane orthogonal to k. Observe that since $(u_k, k) = 0$ we have $\sqcap_k u_k = u_k$. We apply the projection $\sqcap_k$ to (7). The term $p_k k$ disappears and we obtain

$$\frac{du_k}{dt} = -i \sum_{k_1} (u_{k_1}|k) \sqcap_k u_{k-k_1} - \nu k^2 u_k + \sqcap_k f_k \tag{8}$$

The pressure is given by the following formula

$$-i \sum_{k_1} (u_{k_1}|k)(I - \sqcap_k) u_{k-k_1} - i p_k k + (I - \sqcap_k) f_k = 0 \tag{9}$$

Observe that solutions of (8) satisfy incompressibility condition $(u_k, k) = 0$. The the subspace of real functions, which can be defined by $\overline{u_{-k}} = u_k$ for all $k \in \mathbb{Z}^d$, where by $\overline{z}$ for $z \in \mathbb{C}$ we denote the conjugate of z, is invariant under (8). In the sequel we will investigate the equation (8) restricted to this subspace.

Definition 1 *Enstrophy of* $\{u_k, \quad k \in \mathbb{Z}^d\}$ *is*

$$V(\{u_k, \quad k \in \mathbb{Z}^d\}) = \sum_{k \in \mathbb{Z}^d} |k|^2 |u_k|^2$$

3 Construction of trapping regions from [ES, MS]

The idea in [ES, MS] is to construct a trapping region for each Galerkin projection and this trapping region give uniform bounds allowing passing to the limit.

The *trapping region* for an ODE (here Galerkin projection of Navier-Stokes equations) is a set such that the vector field on its boundary is pointing inside, hence no trajectory can leave it in forward time. In the sequel we consider only the Galerkin projection onto the set of modes O, such that if $k \in O$ then $-k \in O$. We will call such projections *symmetric*.

Lemma 1 *$d = 2$. For any solution of (8) (such that all necessary Fourier series converge) or the symmetric Galerkin projection of (8) we have*

$$\frac{dV\{u_k(t)\}}{dt} \leq -2\nu V(\{u_k(t)\}) + 2V(F)\sqrt{V(\{u_k(t)\})}, \tag{10}$$

where $V(F) = \sqrt{\sum |k|^2 f_k^2}$.

The proof can be found in many text-books, see also [Si].

The inequality (10) shows that

$$\frac{dV\{u_k(t)\}}{dt} < 0, \qquad \text{when} \qquad V > V^* = \left(\frac{F}{\nu}\right)^2 \tag{11}$$

Lemma 2 *Assume that $\{u_k, k \in \mathbb{Z}^d\}$ is such that for some $D < \infty$, $\gamma > 1 + \frac{d}{2}$*

$$|u_k| \leq \frac{D}{k^\gamma}, \quad \textit{and} \quad V(\{u_k\}) \leq V_0 \tag{12}$$

then for $d \geq 3$

$$|\sum_{k_1} (u_{k_1}|k) \sqcap_k u_{k-k_1}| \leq \frac{C\sqrt{V_0}D}{k^{\gamma - \frac{d}{2}}}, \tag{13}$$

where constant C depends only on γ and dimension d and for $d = 2$ for any $\epsilon > 0$

$$|\sum_{k_1} (u_{k_1}|k) \sqcap_k u_{k-k_1}| \leq \frac{C(\epsilon,\gamma)\sqrt{V_0}D}{k^{\gamma - \frac{d}{2} - \epsilon}}, \tag{14}$$

Proof:

In order to estimate the sum $|\sum_{k_1} (u_{k_1}|k) \sqcap_k u_{k-k_1}|$ we will use the following inequality

$$|(u_{k_1}|k) \sqcap_k u_{k-k_1}| = |(u_{k_1}|k - k_1) \sqcap_k u_{k-k_1}| \leq |u_{k_1}| \; |k - k_1| \; |u_{k-k_1}| \tag{15}$$

We consider three cases.

Case I. $|k_1| \leq \frac{1}{2}|k|$.

Here $|k - k_1| \geq \frac{1}{2}|k|$ and therefore $|u_{k-k_1}| \; |k - k_1| \leq \frac{D}{|k-k_1|^{\gamma-1}} \leq \frac{2^{\gamma-1}D}{|k|^{\gamma-1}}$. Now observe that

$$\sum_{|k_1| \leq \frac{1}{2}|k|} |u_{k_1}| = \sum_{|k_1| \leq \frac{1}{2}|k|} |k_1| \; |u_{k_1}| \frac{1}{|k_1|} \leq \sqrt{\sum |k_1|^2 |u_{k_1}|^2} \cdot \sqrt{\sum_{|k_1| < \frac{1}{2}|k|} \frac{1}{|k_1|^2}} \tag{16}$$

The sum $\sum_{|k_1|<\frac{1}{2}|k|} \frac{1}{|k_1|^2}$ can be estimates from above by a constant times an integral of $\frac{1}{r^2}$ over the ball of radius $\frac{1}{2}|k|$ with the ball around the origin removed. Hence for $d=2$ we have

$$\sum_{|k_1|\leq\frac{1}{2}|k|} \frac{1}{|k_1|^2} \leq C\int_1^{|k|/2} \frac{rdr}{r^2} \leq C\ln|k| \tag{17}$$

For $d\geq 3$ we have

$$\sum_{|k_1|\leq\frac{1}{2}|k|} \frac{1}{|k_1|^2} \leq C\int_1^{|k|/2} \frac{r^{d-1}dr}{r^2} \leq C|k|^{d-2} \tag{18}$$

From all the above computations it follows that for $d\geq 3$ holds

$$|\sum_{|k_1|\leq\frac{|k|}{2}} (u_{k_1}|k) \sqcap_k u_{k-k_1}| \leq \frac{2^{\gamma-1}D}{|k|^{\gamma-1}}\sqrt{V_0}\sqrt{C}|k|^{\frac{d}{2}-1} = \frac{2^{\gamma-1}D\sqrt{V_0}\sqrt{C}}{|k|^{\gamma-\frac{d}{2}}} \tag{19}$$

For $d=2$ we have

$$|\sum_{|k_1|\leq\frac{|k|}{2}} (u_{k_1}|k) \sqcap_k u_{k-k_1}| \leq \frac{2^{\gamma-1}D}{|k|^{\gamma-1}}\sqrt{V_0}\sqrt{C}\sqrt{\ln|k|} < \frac{C\sqrt{V_0}D}{|k|^{\gamma-1-\epsilon}} \tag{20}$$

Case II. $\frac{1}{2}|k| < |k_1| \leq 2|k|$.

$$|u_{k_1}| < \frac{D}{|k_1^\gamma|} < \frac{D}{\left(\frac{|k|}{2}\right)^\gamma} = \frac{2^\gamma D}{|k|^\gamma} \tag{21}$$

Hence

$$\sum_{\frac{1}{2}|k|<|k_1|\leq 2|k|} |u_{k_1}|\cdot|u_{k-k_1}|\cdot|k-k_1| \leq \frac{2^\gamma D}{|k|^\gamma} \sum_{\frac{1}{2}|k|<|k_1|\leq 2|k|} |u_{k-k_1}|\cdot|k-k_1| \tag{22}$$

We interpret $\sum_{\frac{1}{2}|k|<|k_1|\leq 2|k|} |u_{k-k_1}|\cdot|k-k_1|$ as a scalar product of $|u_{k-k_1}|\cdot|k-k_1|$ and 1, hence by the Schwarz inequality

$$\sum_{\frac{1}{2}|k|<|k_1|\leq 2|k|} |u_{k-k_1}|\cdot|k-k_1| \leq \sqrt{\sum_{|k_1|\leq 3|k|} |u_{k_1}|^2|k_1|^2}\cdot\sqrt{C(3|k|)^d}, \tag{23}$$

where C is such that $C(3|k|)^d$ is greater or equal than the number of vectors in $\mathbb{Z}^d$, which are contained in the ball of radius $3|k|$ around the origin.

Finally we obtain

$$\sum_{\frac{1}{2}|k|<|k_1|\leq 2|k|} |u_{k_1}|\cdot|u_{k-k_1}|\cdot|k-k_1| \leq \frac{2^\gamma D\tilde{C}\sqrt{V_0}}{|k|^{\gamma-\frac{d}{2}}} \tag{24}$$

Case III. $|k_1| > 2|k|$. Here we $|k-k_1| > |k|$.

$$\sum |u_{k_1}||k-k_1||u_{k-k_1}| \le \frac{1}{|k|}\sum |u_{k_1}||k_1||k-k_1||u_{k-k_1}| \le$$

$$\frac{1}{|k|}\sqrt{\sum |u_{k_1}|^2|k_1|^2}\sqrt{\sum |u_{k-k_1}|^2|k-k_1|^2} \le \frac{\sqrt{V_0}}{|k|}\sqrt{\sum_{|k_1|>2|k|}\frac{D^2}{|k_1|^{2\gamma-2}}} =$$

$$\frac{\sqrt{V_0}D}{|k|}\sqrt{\sum_{|k_1|>2|k|}\frac{1}{|k_1|^{2\gamma-2}}}$$

To estimate $\sum_{|k_1|>2|k|}\frac{1}{|k_1|^{2\gamma-2}}$ observe that we have (we denote all constant factors depending on γ by C)

$$\sum_{|k_1|>2|k|}\frac{1}{|k_1|^{2\gamma-2}} \le C\int_{|k_1|>2|k|}\frac{1}{|k_1|^{2\gamma-2}}d^dk_1 = C\int_{2|k|}^{\infty}\frac{1}{r^{2\gamma-2}}r^d dr =$$

$$C\int_{2|k|}^{\infty} r^{-(2\gamma-2-d+1)} = C|k|^{-2\gamma-2-d}$$

Observe that we used here the assumption $\gamma > 1+\frac{d}{2}$, which guarantees that $2\gamma-2-d+1>1$ so the integral converges.

Hence for the case III we obtain

$$\sum \dots \le \frac{\sqrt{V_0}DC}{|k|^{\gamma-\frac{d}{2}}} \tag{25}$$

Adding cases I,II,III we obtain for $d \ge 3$

$$|\sum_{k_1}(u_{k_1}|k)\sqcap_k u_{k-k_1}| \le \frac{C\sqrt{V_0}D}{|k|^{\gamma-\frac{d}{2}}} \tag{26}$$

For $d=2$ we obtain

$$|\sum_{k_1}(u_{k_1}|k)\sqcap_k u_{k-k_1}| \le \frac{C\sqrt{V_0}D}{|k|^{\gamma-\frac{d}{2}-\epsilon}} \tag{27}$$

∎

Lemma 3 *Assume that $\gamma > d$, then*

$$\sum_{k_1\in\mathbb{Z}^d\setminus\{0,k\}}\frac{1}{|k_1|^\gamma|k-k_1|^\gamma} \le \frac{C_Q(d,\gamma)}{|k|^\gamma} \tag{28}$$

Proof: We consider three cases.
Case I. $|k_1| < \frac{|k|}{2}$, hence $|k-k_1| \ge \frac{|k|}{2}$.

We have

$$\sum_{|k_1|<\frac{|k|}{2}} \leq \sum_{|k_1|<\frac{|k|}{2}} \frac{1}{k_1^\gamma}\frac{2^\gamma}{|k|^\gamma} < \frac{2^\gamma}{|k|^\gamma} C \int_1^\infty \frac{r^{d-1}}{r^\gamma} dr$$

The improper integral $\int_1^\infty \frac{r^{d-1}}{r^\gamma} dr$ converges, because $\gamma > d$.

Hence

$$\sum_{|k_1|<\frac{|k|}{2}} < \frac{C_I(d,\gamma)}{|k|^\gamma}$$

Case II. $\frac{|k|}{2} < |k_1| \leq 2|k|$.

$$\sum_{\frac{|k|}{2}<|k_1|\leq 2|k|} \leq \frac{2^\gamma}{|k|^\gamma} \sum_{\frac{|k|}{2}<|k_1|\leq 2|k|} \frac{1}{|k-k_1|^\gamma} <$$

$$\frac{2^\gamma}{|k|^\gamma} \sum_{|k_1|\leq 3|k|} \frac{1}{|k_1|^\gamma} < \frac{2^\gamma}{|k|^\gamma} C \int_1^\infty \frac{r^{d-1}}{r^\gamma} dr$$

Hence

$$\sum_{\frac{|k|}{2}<|k_1|\leq 2|k|} < \frac{C_{II}(d,\gamma)}{|k|^\gamma}$$

Case III. $2|k| < |k_1|$, hence $|k-k_1| > |k|$.

$$\sum_{2|k|<|k_1|} < \frac{1}{|k|^\gamma} \sum \frac{1}{|k_1|^\gamma} < \frac{C_{III}(d,\gamma)}{|k|^\gamma}$$

∎

3.1 The construction of the trapping region I.

We take $V_0 > V^*$, $\gamma \geq 2.5$ and K such that $f_k = 0$ for $|k| > K$. We set

$$N(V_0, K, \gamma, D) = \left\{ \{u_k\} \mid V(\{u_k\}) \leq V_0, \qquad |u_k| \leq \frac{D}{|k|^\gamma}, \quad |k| > K \right\} \tag{29}$$

We prove that

Theorem 4 *Let $d = 2$ and $C(\gamma)$ be a constant from lemma 2. If $K > \frac{C^2 V_0}{\nu^2}$ and $D > \sqrt{V_0} K^{\gamma-1}$, then $N = N(V_0, K, \gamma, D)$ is a trapping region for each Galerkin projection.*

Proof: Observe that for $D \geq \sqrt{V_0} K^{\gamma-1}$ for all $\{u_k\} \in N$ holds

$$|u_k| \leq \frac{D}{|k|^\gamma}. \tag{30}$$

To prove this observe that (30) holds for $|k| > K$ by the definition of N. For $|k| \leq K$ we proceed as follows: since $V(\{u_k\}) \leq V_0$ then $|k|^2|u_k|^2 \leq V_0$. So we have

$$|u_k| \leq \frac{\sqrt{V_0}}{|k|} \leq \frac{D}{|k|^\gamma}, \quad |k| \leq K \tag{31}$$

for D such that $\sqrt{V_0}|k|^{\gamma-1} \leq D$ for all $|k| \leq K$.

We will show now that on the boundary of N (we are considering the Galerkin projection) the vector field is pointing inside. For points $V(\{u_k\}) = V_0$ it follows from (11). For points such that $u_k = \frac{D}{|k|^\gamma}$ for some $|k| > K$ we have from lemma 2 (with $\epsilon = 1/2$)

$$\frac{d|u_k|}{dt} \leq \frac{C\sqrt{V_0}D}{|k|^{\gamma-\frac{3}{2}}} - \nu|k|^2\frac{D}{|k|^\gamma} < 0, \tag{32}$$

which is satisfied when

$$C\sqrt{V_0} < \nu|k|^{1/2}. \tag{33}$$

Observe that (33) holds for $|k| \geq K$ if $K > \frac{C^2V_0}{\nu^2}$.

∎

Remark 1 *Observe that in the proof it was of crucial importance that the constant D entered linearly in the estimate in lemma 2 and due to this fact did not appear in (33). For example assume that the estimate of the nonlinear part will be of the form $\frac{D^2C}{|k|^{\gamma-\frac{3}{2}}}$ then instead of (33) we will have*

$$CD < \nu|k|^{1/2}$$

which will require that $K > \frac{C^2D^2}{\nu^2}$, which might be incompatible with $D > \sqrt{V_0}K^{\gamma-1}$.

This shows how important it was to use the enstrophy in these estimates.

3.2 The construction of the trapping region II - exponential decay

Theorem 5 *Assume that $\gamma \geq 2.5$, $d = 2$. Then the set*

$$N_e = N(V_0, K, \gamma, D) \cap \left\{ \{u_k\} \mid |u_k| \leq \frac{D_2}{|k|^\gamma} e^{-a|k|} \text{ for } |k| > K_e \right\}, \tag{34}$$

where $N(V_0, K, \gamma, D)$ is a trapping region from theorem 4, $D_2 > D$, $K_e > \frac{C_Q(d,\gamma)D_2}{\nu}$ (C_Q was obtained in lemma 3) and $0 < a < \frac{1}{K_e} \ln \frac{D_2}{D}$ is a trapping region for each Galerkin projection.

Proof: The set N_e constructed so that for all $|k| \leq K_e$ the trapping (the vector field is pointing toward the interior of N_e on the boundary) is obtained from $N(V_0, K, \gamma, D)$ and for $|k| > K_e$ it results from the new exponential estimates.

Observe that a is such that $\frac{D_2}{|k|^\gamma}e^{-a|k|} > \frac{D}{|k|^\gamma}$ for all $|k| \leq K_e$. This solves the trapping for $|k| \leq K_e$.

Hence to prove the trapping it is enough to consider the boundary points such that $|u_k| = \frac{D_2}{|k|^\gamma}e^{-a|k|}$ for some $k > K_e$. For such a point and $|k|$ we have

$$\frac{d|u_k|}{dt} \leq \left|\sum(u_{k_1}|k) \sqcap_k u_{k-k_1}\right| - \nu|k|^2|u_k| \leq$$

$$\sum |u_{k-1}||k||u_{|k-k_1|}| - \nu|k|^2|u_k| \leq D_2^2|k| \sum \frac{e^{-a|k_1|}e^{-a|k-k_1|}}{|k_1|^\gamma|k-k_1|^\gamma} - \nu|k|^2|u_k|$$

Observe that $e^{-a|k_1|}e^{-a|k-k_1|} \leq e^{-a|k|}$. From this and lemma 3 we obtain

$$\frac{d|u_k|}{dt} < \frac{D_2^2 C_Q(\gamma,d)}{|k|^{\gamma-1}}e^{-a|k|} - \nu|k|^2|u_k|$$

Hence $\frac{d|u_k|}{dt} < 0$, when

$$|u_k| = \frac{D_2}{|k|^\gamma}e^{-a|k|} > \frac{C_Q D_2^2}{\nu|k|^{\gamma+1}}e^{-a|k|}$$

Which is equivalent to

$$|k| > K_e = \frac{C_Q D_2}{\nu}.$$

∎

3.3 Trapping region III - exponential decay in time

Theorem 6 *Let $t_0 > 0$. Assume that $\gamma \geq 2.5$, $d = 2$. Then the set*

$$N_e = N(V_0, K, \gamma, D) \cap \left\{ \{u_k\} \mid |u_k| \leq \frac{D_3}{|k|^\gamma}e^{-a_3|k|t} \text{ for } |k| > K_e \right\}, \qquad (35)$$

where $N(V_0, K, \gamma, D)$ is a trapping region from theorem 4, $D_3 > D$, $K_e > \frac{D_3 C_Q(d,\gamma)}{\nu}$ (C_Q was obtained in lemma 3) and $0 < a_3 < \frac{1}{K_e t_0}\ln\frac{D_3}{D}$ is a trapping region for each Galerkin projection for $0 \leq t \leq t_0$.

Proof: The set N_e constructed so that for all $|k| \leq K_e$ the trapping is obtained from $N(V_0, K, \gamma, D)$ and for $|k| > K_e$ it results from the new exponential estimates.

To be sure that the boundary of N_e for $|k| < K_e$ is obtained from $N(V_0, K, \gamma, D)$ we require that

$$\frac{D}{|k|^\gamma} < \frac{D_3}{|k|^\gamma}e^{-a_3|k|t}, \quad \text{for } 0 \leq t \leq t_0 \text{ and } |k| \leq K_e. \qquad (36)$$

Easy computations show that (36) holds iff $a_3 < \frac{1}{K_e t_0}\ln\frac{D_3}{D}$.

To have the trapping for $|k| > K_e$ we need to show that $\frac{d|u_k|}{dt} < 0$ if $|u_k| = \frac{D_3}{|k|^\gamma} e^{-a_3 t}$, for some $0 \leq t \leq t_0$ and $|k| > K_e$).

$$\begin{aligned} \frac{d|u_k|}{dt} &\leq \sum |u_{k_1}||k||u_{k-k_1}| - \nu|k|^2|u_k| \leq \\ |k|D_3^2 \sum \frac{e^{-a_3|k_1|t} e^{-a_3|k-k_1|t}}{|k_1|^\gamma |k-k_1|^\gamma} &- \nu|k|^2|u_k| \leq \\ |k|e^{-a_3|k|t} D_3^2 \sum \frac{1}{|k_1|^\gamma |k-k_1|^\gamma} &- \nu|k|^2|u_k| \leq \\ \frac{e^{-a_3|k|t} D_3^2 C_Q(d,\gamma)}{|k|^{\gamma-1}} &- \nu|k|^2|u_k| \end{aligned}$$

Hence $\frac{d|u_k|}{dt} < 0$ if

$$\frac{D_3^2 C_Q(d,\gamma)}{\nu|k|^{\gamma+1}} e^{-a_3|k|t} < |u_k| = \frac{D_3}{|k|^\gamma} e^{-a_3|k|t}, \tag{37}$$

which is equivalent to

$$\frac{D_3 C_Q}{\nu} < |k|. \tag{38}$$

Hence for $K_e \geq \frac{D_3 C_Q}{\nu}$ we obtain the trapping. ∎

4 Passing to the limit for Galerkin projections via Ascoli-Arzela lemma

The goal of this section is a relatively simple argument for the passing to the limit with Galerkin projections.

All what follows was essentially proved in [ZM]. We will also use some conventions used there.

Let H be a Hilbert space. Let $\phi_1, \phi_2, \ldots$ be a orthonormal basis in H.

Let $A_n : H \to H$ be denote the projection onto 1-dimensional subspace $\langle \phi_n \rangle$, i.e $x = \sum A_n(x)\phi_n$ for all $x \in H$. By V_n we will denote the space spanned by $\{\phi_1, \ldots, \phi_n\}$. Let P_n denote the projection onto , $Q_n = I - P_n$.

Definition 2 *Let $W \subset H$ and $F : dom(H) \to H$. We say that W and F satisfy conditions C1,C2,C3 if*

C1 *There exists $M \geq 0$, such that $P_n(W) \subset W$ for $k \geq M$*

C2 *Let $\hat{u_k} = \max_{x \in W} |A_k x|$. Then, $\hat{u} = \sum \hat{u_k} \in H$. In particular, $|\hat{u}| < \infty$.*

C3 *The function $x \mapsto F(x)$ is continuous on W. The sequence $f = \{f_k\}$, given by $f_k = \max_{x \in W} |A_k F(x)|$ is in H. In particular, $|f| < \infty$.*

Observe that condition C2 implies that the set W is compact. Conditions C2 and C3 guarantee good behavior of F with respect to passage the limit. We have here continuous function on the compact set, this is also perfect setting for study the dynamics of $x' = F(x)$ (see [ZM] for more details).

Lemma 7 *Assume that $W \subset H$ and F satisfy C1,C2,C3. Let $x : [0,T] \to W$ be such that for each n*

$$\frac{dA_n x}{dt} = A_n(F(x)). \tag{39}$$

Then

$$x' = F(x). \tag{40}$$

Proof: Let us set $x_k = A_k x$. Let us fix $\epsilon > 0$ and $t \in [0,T]$. For any n we have

$$\left| \frac{x(t+h) - x(t)}{h} - F(x) \right| \leq \left| \frac{P_n x(t+h) - P_n x(t)}{h} - P_n F(x) \right| + \tag{41}$$

$$\left| \frac{1}{h} \sum_{k=n+1}^{\infty} (x_k(t+h) - x_k(t)) \phi_k \right| + |Q_n F(x)| \tag{42}$$

We will estimate the three terms on the right hand side separately. From **C3** it follows for a given $\epsilon > 0$ there exists n_0 such that $n > n_0$ implies

$$|Q_n(F(a))| < \epsilon/3.$$

From now on fix $n > n_0$. Again **C3** and the mean value theorem implies

$$\begin{aligned} \left| \sum_{k=n+1}^{\infty} \frac{1}{h}(x_k(t+h) - x_k(t)) \phi_k \right| &= \left| \sum_{k=n+1}^{\infty} \frac{dx_k}{dt}(t + \theta_k h) \phi_k \right| \\ &\leq \left| \sum_{k=n+1}^{\infty} f_k \phi_k \right| < \epsilon/3. \end{aligned}$$

Finally, for h sufficiently small,

$$\left| \frac{1}{h}(P_n x(t+h) - P_n x(t)) \phi_k - P_n F(x) \right| < \epsilon/3$$

and hence the desired limit is obtained.

∎

Lemma 8 *Assume that $W \subset H$ and the function F satisfy C1,C2,C3. Let $x_0 \in W$. Assume that for each n a function $x_n : [0,T] \to P_n(W)$ is a solution of the problem (Galerkin projection of $x' = F(x)$)*

$$x_n' = P_n(F(x)), \quad x_n(0) = P_n(x_0). \tag{43}$$

Assume also that x_n converges uniformly to $x^* : [0,T] \to W$.

Then x^* solves the following initial value problem

$$x' = F(x), \quad x(0) = x_0 \tag{44}$$

Proof: We show first that for all n and $t \in [0,T]$ holds

$$P_n x^*(t) = P_n x_0 + \int_0^t P_n F(x^*(s))ds. \tag{45}$$

Let us fix n. Observe that for each $m \geq n$ the following equation holds

$$P_n x_m(t) = P_n x_0 + \int_0^t P_n F(x_m(s))ds \tag{46}$$

Since the series x_m converges uniformly to x^*, then also $P_n x_m$ converges uniformly to $P_n x^*$. Observe that also the functions $P_n F(x_m)$ converge uniformly to $P_n F(x^*)$ as the composition of the uniformly continuous function $P_n F$ (because F is a continuous function on the compact set W) with a uniformly convergent sequence, hence also the integral in (46) is converging (uniformly in $t \in [0,T]$) to $\int_0^t P_n F(x^*(s))$. This proves (45). Differentiation of (45) gives

$$\frac{dP_n x^*}{dt} = P_n F(x^*). \tag{47}$$

The assertion follows from lemma 7 ∎

Theorem 9 *Assume that $W \subset H$ and the function F satisfy C1,C2,C3. Let $x_0 \in W$. Assume that for each n a function $x_n : [0,T] \to P_n(W)$ is a solution of the problem (Galerkin projection of $x' = F(x)$)*

$$x_n' = P_n(F(x)), \quad x_n(0) = P_n(x_0). \tag{48}$$

Then there exists $x^* : [0,T] \to W$, such that x^* solves the following initial value problem

$$x' = F(x), \quad x(0) = x_0 \tag{49}$$

Proof: The idea goes as follows, we would like to pickup a convergent subsequence from $\{x_n\}$ using Ascoli-Arzela compactness theorem. Later we show that the limit function x^* solves (49).

Observe first that due to compactness of W and since $x_n(t) \in W$ for $t \in [0,T]$ the sequence $\{x_n\}$ is contained in a compact set. Observe that the derivatives $x_n'(t)$ are uniformly bounded by $|F(W)|$, hence the sequence of functions x_n is equicontinuous. From Ascoli-Arzela theorem if follows that there exists a subsequence converging uniformly to $x^* : [0,T] \to W$. Without loss of generality we can assume that the whole sequence x_n is converging uniformly to x^*. It is obvious that $x^*(0) = x_0$. The assertion of the theorem follows from lemma 8. ∎

5 Passing to the limit, an analytic argument

The goal of this section is to present another argument for the limit of Galerkin projections. Compared to section 4 we assume more about the function F

and we add a new condition D on the trapping regions, which are satisfied for the Navier-Stokes system and the trapping regions constructed in section 3. We obtain better results about convergence plus uniqueness and Lipschitz constants for the induced flow.

We will use here the notations introduced in section 4. We investigate the Galerkin projections of the following problem

$$x' = F(x) = L(x) + N(x), \tag{50}$$

where L is a linear operator and N is a nonlinear part of F. We assume that the basis $\phi_1, \phi_2, \dots$ of H is build from eigenvectors of L. We assume that the corresponding eigenvalues λ_k (i.e. $L\phi_k = \lambda_k \phi_k$) can ordered so that

$$\lambda_1 \geq \lambda_2 \geq \dots, \quad \text{and} \quad \lim_{k\to\infty} \lambda_k = -\infty.$$

Hence we can have only a finite number of positive eigenvalues.

5.1 Estimates based on logarithmic norms

The goal of this section is to recall some results about one-sided Lipschitz constants of the flows induced by ODE's. We will invoked here results from [HNW].

Definition 3 *[HNW, Def. I.10.4] Let Q be a square matrix, then we call*

$$\mu(A) = \lim_{h>0, h\to 0} \frac{\|I + hQ\| - 1}{h}$$

the logarithmic norm *of Q.*

Theorem 10 *[HNW, Th. I.10.5] The logarithmic norm is obtained by the following formulas*

- *for Euclidean norm*

$$\mu(Q) = \textit{the largest eigenvalue of} \quad 1/2(Q + Q^T).$$

- *for max norm $\|x\| = \max_k |x_k|$*

$$\mu(Q) = \max_k \left(q_{kk} + \sum_{i\neq k} |q_{ki}| \right)$$

- *for norm $\|x\| = \sum_k |x_k|$*

$$\mu(Q) = \max_i \left(q_{ii} + \sum_{k\neq i} |q_{ki}| \right)$$

Consider now the differential equation

$$x' = f(x), \quad f \in C^1. \tag{51}$$

Let $\varphi(t, x_0)$ we denote the solution of equation (51) with the initial condition $x(0) = x_0$. By $\|x\|$ we denote a fixed arbitrary norm in $\mathbb{R}^n$.

The following theorem was proved in [HNW, Th. I.10.6] (for nonautonomous ODE, here we restrict ourselves to an autonomous case only and we use a different notation)

Theorem 11 *Let $y : [0, T] \to \mathbb{R}^n$ and $\varphi(\cdot, x_0)$ is defined for $t \in [0, T]$. Suppose that we have the following estimates*

$$\mu\left(\frac{\partial f}{\partial x}(\eta)\right) \leq l(t), \quad \textit{for } \eta \in [y(t), x(t)]$$

$$\left\|\frac{dy}{dt}(t) - f(y(t))\right\| \leq \delta(t), \quad \|y(0) - x_0\| \leq \rho.$$

Then for $0 \leq t \leq T$ we have

$$\|\varphi(t, x_0) - y(t)\| \leq e^{L(t)} \left(\rho + \int_0^t e^{-L(s)} \delta(s) ds\right),$$

where $L(t) = \int_0^t l(s) ds$.

From the above theorem one easily derives the following

Lemma 12 *Let $y : [0, T] \to \mathbb{R}^n$ and $\varphi(\cdot, x_0)$ is defined for $t \in [0, T]$. Suppose that Z is a convex set such that we have the following estimates*

$$y([0, T]), \varphi([0, T], x_0) \in Z$$

$$\mu\left(\frac{\partial f}{\partial x}(\eta)\right) \leq l, \quad \textit{for } \eta \in Z$$

$$\left\|\frac{dy}{dt}(t) - f(y(t))\right\| \leq \delta, \quad \|y(0) - x_0\| \leq \rho.$$

Then for $0 \leq t \leq T$ we have

$$\|\varphi(t, x_0) - y(t)\| \leq e^{lt}\rho + \delta \frac{e^l t - 1}{l}, \quad \textit{if } l \neq 0.$$

For $l = 0$ we have

$$\|\varphi(t, x_0) - y(t)\| \leq \rho + \delta t.$$

5.2 Application to Galerkin projections - uniqueness and another proof of convergence

Definition 4 *We say that $W \subset H$ and $F = N + L$ satisfy condition* D *if the following condition holds*

D *there exists $l \in \mathbb{R}$ such that for all $k = 1, 2, \ldots$*

$$1/2 \sum_{i=1}^{\infty} \left| \frac{\partial N_k}{\partial x_i} \right| (W) + 1/2 \sum_{i=1}^{\infty} \left| \frac{\partial N_i}{\partial x_k} \right| (W) + \lambda_k \leq l \tag{52}$$

The main idea behind the condition **D** is the ensure that the logarithmic norms for all Galerkin projections are uniformly bounded.

Theorem 13 *Assume that $W \subset H$ and F satisfy conditions* C1,C2,C3,D *and W is convex. Assume that $P_n(W)$ is a trapping region for the n-dimensional Galerkin projection of (50) for all $n > M_1$. Then*

1. **Uniform convergence and existence** *For a fixed $x_0 \in W$, let $x_n : [0, \infty] \to P_n(W)$ be a solution of $x' = P_n(F(x))$, $x(0) = P_n x_0$. Then x_n converges uniformly on compact intervals to a function $x^* : [0, \infty] \to W$, which is a solution of (50) and $x^*(0) = x_0$*

2. **Uniqueness within** W. *There exists only one solution of the initial value problem (50), $x(0) = x_0$ for any $x_0 \in W$, such that $x(t) \in W$ for $t > 0$.*

3. **Lipschitz constant**. *Let $x : [0, \infty] \to W$ and $y : [0, \infty] \to W$ be solutions of (50), then*
$$|y(t) - x(t)| \leq e^{lt} |x(0) - y(0)|$$

4. **Semidynamical system.** *The map $\varphi : \mathbb{R}_+ \times W \to W$, where $\varphi(\cdot, x_0)$ is a unique solution of equation (50), such that $\varphi(0, x_0) = x_0$ defines a semidynamical system on W, namely*
   - *φ is continuous*
   - $\varphi(0, x) = x$
   - $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$

Proof: By $|x|_n$ we will denote $|P_n(x)|$, i.e. Euclidean norm in $\mathbb{R}^n$.

Let
$$\delta_n = \max_{x \in W} |P_n(F(x)) - P_n(F(P_n x))|.$$

Obviously $\delta_n \to 0$ for $n \to \infty$, because F is uniformly continuous on W and $P_n W \subset W$, for $n \geq M$.

Let us consider the logarithmic norm of the vector field for the n-dimensional Galerkin projection. We will estimate it using the euclidean norm on $P_n H = \mathbb{R}^n$(which coincides with the norm inherited from H). Since

$$\left[\frac{\partial P_n(L + N)}{\partial(x_1 \ldots x_n)} \right]_{ij} = \frac{\partial N_i}{\partial x_j} + \delta_{ij} \lambda_j, \tag{53}$$

we need to estimate the largest eigenvalue of the following matrix, $Q_n(x)$ for $x \in P_n(W)$,

$$Q_{n,ij}(x) = \frac{1}{2} \frac{\partial N_i}{\partial x_j}(x) + \frac{1}{2} \frac{\partial N_j}{\partial x_i}(x) + \delta_{ij} \lambda_j, \quad \text{for } i, j = 1, \ldots, n \tag{54}$$

where δ_{ij} is a Kronecker symbol, i.e. $\delta_{ij} = 1$, if $i = j$ and $\delta_{ij} = 0$ otherwise.

To estimate the largest eigenvalue of Q_n we will use the Gershgorin theorem (see [QSS, Property 5.2]), which states that all eigenvalues of a square $n \times n$-matrix A, $\sigma(A)$, satisfy

$$\sigma(A) \subset \cup_{j=1}^{n} \{ z \in \mathbb{C} : \quad |z - A_{jj}| < \Sigma_{i=1, i\neq j} |A_{ij}| \}. \tag{55}$$

From above equation and condition D it follows immediately that eigenvalues of Q_n are less then or equal to l_n, where

$$l_n = \max_{k=1,\dots,n} \max_{x \in P_n W} \sum_{i=1}^{n} \left(1/2 \left| \frac{\partial N_k}{\partial x_i}(x) \right| + 1/2 \left| \frac{\partial N_i}{\partial x_k}(x) \right| \right) + \lambda_k. \tag{56}$$

From the assumption **D**, it follows that we have uniform bound on l_n, namely

$$l_n \leq l, \quad \text{for all } n. \tag{57}$$

Let us take $m \geq n$. Let $x_n : [0,T] \to P_n W$ and $x_m : [0,T] \to P_m W$ be the solutions of n- and m-dimensional projections of (50). From lemma 12 it follows immediately that (we treat here $P_n x_m$ as a perturbed 'solution' y)

$$|x_n(t) - P_n(x_m(t))|_n \leq e^{lt} |x_n(0) - P_n x_m(0)| + \delta_n \frac{e^{lt} - 1}{l} \tag{58}$$

To prove uniform convergence of $\{x_n\}$ starting from the same initial condition observe that

$$|x_n(t) - x_m(t)| \leq |x_n(t) - P_n(x_m(t))|_n + |(I - P_n) x_m(t)| \leq$$
$$\delta_n \frac{e^{lt} - 1}{l} + |(I - P_n) x_m(t)| \leq \delta_n \frac{e^{lT} - 1}{l} + |(I - P_n) W|$$

This shows that $\{x_n\}$ is a Cauchy sequence in $\mathcal{C}([0,T], H)$, hence it converges uniformly to $x^* : [0,T] \to W$. From lemma 8 it follows that $\frac{dx^*}{dx} = F(x)$.

Uniqueness. Let $x : [0,T] \to W$ be a solution of (50) with the initial condition $x(0) = x_0$. We will show that x_n converge to x. We apply lemma 12 to n-dimensional projection and the function $P_n x(t)$. We obtain

$$|x_n(t) - P_n(x(t))|_n \leq \delta_n \frac{e^{lt} - 1}{l}. \tag{59}$$

Since the tail $(I - P_n)x(t)$ is uniformly bounded we see that $x_n \to x$ uniformly.

Lipschitz constant on W. From equation (58) applied to $m = n$ for different initial conditions (we denote the functions by x_n and y_n and the initial conditions x_0 and y_0) we obtain

$$|x_n(t) - y_n(t)| \leq e^{lt} |P_n x_0 - P_n y_0| + \delta_n \frac{e^{lt} - 1}{l} \tag{60}$$

Let $x_n \to x$ and $y_n \to y$. Then passing to the limit in (60) gives

$$|x(t) - y(t)| \leq e^{lt} |x_0 - y_0|. \tag{61}$$

Assertion 4 follows easily from from the previous ones. ∎

6 Existence theorems for Navier-Stokes system in 2D

6.1 Some easy lemmas about Fourier series

Lemma 14 *Let $u \in C^n(\mathbb{T}^d, \mathbb{C})$ and let u_k for $k \in \mathbb{Z}^d$ be a Fourier coefficient of u. Then there exists M*

$$|u_k| \leq \frac{M}{|k|^n}$$

Lemma 15 *Assume that $|u_k| \leq \frac{M}{|k|^\gamma}$ for $k \in \mathbb{Z}^d$. Let $n \in \mathbb{N}$ be such that $\gamma - n > d$, then the function $u(x) = \sum_{k\in\mathbb{Z}^d} u_k e^{ikx}$ belongs to $C^n(\mathbb{T}^d, \mathbb{C})$. The series*

$$\frac{\partial^s u}{\partial x_{i_1} \dots x_{i_s}} = \sum_{k\in\mathbb{Z}^d} u_k \frac{\partial^s}{\partial x_{i_1} \dots x_{i_s}} e^{ikx}$$

converge uniformly for $0 \leq s \leq n$.

Lemma 16 *Assume that for some $\gamma > 0$, $a > 0$ and $D > 0$ we have $|u_k| \leq \frac{De^{-a|k|}}{|k|^\gamma}$ for $k \in \mathbb{Z}^d \setminus \{0\}$.*

Then the function $u(x) = \sum_{k\in\mathbb{Z}^d} u_k e^{ikx}$ is analytic.

Let $H = \left\{ \{u_k\} \mid \sum_{k\in\mathbb{Z}^d} |u_k|^2 < \infty \right\}$. Obviously H is a Hilbert space. Let F be the right side of (8)

$$F(u)_k = -i \sum_{k_1} (u_{k_1}|k) \sqcap_k u_{k-k_1} - \nu k^2 u_k + \sqcap_k f_k \tag{62}$$

For a general $u \in H$ we cannot claim that $F(u) \in H$. But when $|u_k|$ decreases fast enough we have the following

Lemma 17 *Let $W(D, \gamma) = \left\{ u \in H \mid |u_k| \leq \frac{D}{|k|^\gamma} \right\}$, then*

1. *if $\gamma > \frac{d}{2}$, then $W(D, \gamma)$ satisfies condition C2.*

2. *if $\gamma - 2 > \frac{d}{2}$ and $\gamma > d$, then the function $F : W(D, \gamma) \to H$ is continuous and condition C3 is satisfied on $W(D, \gamma)$.*

3. *if $\gamma > d + 1$, then condition D is satisfied on $W(D, \gamma)$.*

Proof: To prove assertion 1 it is enough to show that $W(d, \gamma)$ is bounded, closed (obvious) and is is componentwise bounded by some $v = \{v_k\}$, such that $v \in H$. We set $v_k = \frac{D}{|k|^\gamma}$. Observe that $v \in H$, because

$$\sum_{k\in\mathbb{Z}^d} |v_k|^2 \leq CD^2 \sum_{n=1}^{\infty} \frac{n^{d-1}}{n^{2\gamma}} \tag{63}$$

and the series converge when $2\gamma - (d-1) > 1$. This concludes the proof of assertion 1.

To prove assertion 2 we can assume that $f = 0$ (it is just a constant vector in H). From lemma 3 if follows immediately that for $u \in W$ we obtain

$$|F(u)_k| \leq \frac{C}{|k|^{\gamma-1}} + \frac{\nu D}{|k|^{\gamma-2}} \leq \frac{B}{|k|^{\gamma-2}}.$$

Hence $F(u) \in W(B, \gamma-2) \subset H$, when $\gamma - 2 > \frac{d}{2}$. Hence $F(W(D,\gamma)) \subset W(B, \gamma-2)$. Since the convergence in $W(B,\gamma-2)$ is equivalent to componentwise convergence, the same holds for the continuity. It is obvious that $F(u)_k$ continuous on $W(d,\gamma)$, because the series defining it is uniformly convergent, hence F is continuous on $W(d,\gamma)$.

We prove now assertion 3. Observe that

$$\frac{\partial N_k}{\partial u_{k_1}} = (\cdot|k) \sqcap_k u_{k-k_1} + (u_{k-k_1}|k)\sqcap_k \tag{64}$$

We will treat here u_k as one dimensional object, but the argument is generally correct, i.e. treating u_k as a vector will introduce only an additional constant and will not affect the proof. We estimate

$$\left|\frac{\partial N_k}{\partial u_{k_1}}\right|(W) \leq \frac{2D|k|}{|k-k_1|^\gamma} \tag{65}$$

Hence the sum, $S(k)$, appearing in condition D can be estimated as follows

$$S(k) = 1/2 \sum_{k_1 \in \mathbb{Z}^d \setminus \{0,k\}} \left|\frac{\partial N_k}{\partial u_{k_1}}\right|(W) + 1/2 \sum_{k_1 \in \mathbb{Z}^d \setminus \{0,k\}} \left|\frac{\partial N_{k_1}}{\partial u_k}\right|(W) \leq$$
$$D|k| \sum_{k_1 \in \mathbb{Z}^d \setminus \{0,k\}} \frac{1}{|k-k_1|^\gamma} + D \sum_{k_1 \in \mathbb{Z}^d \setminus \{0,k\}} \frac{|k_1|}{|k-k_1|^\gamma}$$

Now observe that

$$\sum_{k_1 \in \mathbb{Z}^d \setminus \{0,k\}} \frac{1}{|k-k_1|^\gamma} < \sum_{k_1 \in \mathbb{Z}^d, k_1 \neq 0} \frac{1}{|k|^\gamma} = C(d,\gamma) < \infty, \quad \text{for } \gamma > d \tag{66}$$

To estimate the sum $\sum_{k_1 \in \mathbb{Z}^d \setminus \{0,k\}} \frac{|k_1|}{|k-k_1|^\gamma}$ we show that there exists a constant A, such that

$$\frac{|k_1|}{|k-k_1|} < A|k|, \quad \text{for } k, k_1 \in \mathbb{Z}^d \setminus \{0\},\ k \neq k_1. \tag{67}$$

Observe that for $|k_1| \leq 2|k|$, $k_1 \neq 0$, $k_1 \neq k$ we can estimate the denominator by 1 hence we have

$$\frac{|k_1|}{|k-k_1|} \leq 2|k|. \tag{68}$$

For $|k_1| > 2|k|$ we have

$$\frac{|k_1|}{|k-k_1|} = \frac{1}{\left|\frac{k_1}{|k_1|} - \frac{k}{|k_1|}\right|} \leq \frac{1}{1 - \frac{|k|}{|k_1|}} \leq 2. \tag{69}$$

So we can take $A = 2$.

Now we make the following estimate

$$\sum_{k_1 \in \mathbb{Z}^d \setminus \{0,k\}} \frac{|k_1|}{|k-k_1|^\gamma} \leq A|k| \sum_{k_1 \in \mathbb{Z}^d \setminus \{0,k\}} \frac{1}{|k-k_1|^{\gamma-1}} < AC(d,\gamma-1)|k|, \tag{70}$$

provided $\gamma - 1 > d$.

So we have $S(k) < (DC(d,\gamma) + ADC(d,\gamma-1))\,|k|$ and since $\lambda_k = -\nu|k|^2$, we see that l satisfying condition D exists. ∎

6.2 Existence theorems

We set the dimension $d = 2$. We again assume that the force f is such that $f_k = 0$ for $|k| > K$ (in [MS]) more general force is treated.

Observe that from lemma 17 it follows that to have conditions C1, C2, C3, D on the trapping regions constructed in section 3 we need $\gamma > 3$.

Theorem 18 *If for some D and $\gamma > 3$*

$$|u_k(0)| \leq \frac{D}{|k|^\gamma} \tag{71}$$

then the solution of (8) is defined for all $t > 0$ and there exists a constant D', such that

$$|u_k(t)| \leq \frac{D'}{|k|^\gamma}, \quad t > 0. \tag{72}$$

The following theorem tells that if we start with analytic initial conditions that the solution will remain analytic (in space variables).

Theorem 19 *If for some D, $\gamma > 3$ and $a > 0$*

$$|u_k(0)| \leq \frac{D}{|k|^\gamma} e^{-a|k|} \tag{73}$$

then the solution of (8) is defined for all $t > 0$ and there exist constants D' and $a' > 0$ such that

$$|u_k(t)| \leq \frac{D'}{|k|^\gamma} e^{-a'|k|}, \quad t > 0 \tag{74}$$

Next theorem states that the solution starting from regular initial conditions becomes analytic immediately.

Theorem 20 *Assume that for some D, $\gamma > 3$ and $a > 0$ the initial conditions satisfy*

$$|u_k(0)| \leq \frac{D}{|k|^\gamma} \tag{75}$$

then the solution of (8) is defined for all $t > 0$ and and for any $t_0 > 0$ one can find constants D' and $a' > 0$ such that

$$|u_k(t_0)| \leq \frac{D'}{|k|^\gamma} e^{-a'|k|} \tag{76}$$

Proof of theorem 18: Observe first that the enstrophy of $\{u_k(0)\}$ is finite. Let take $V_0 > \max(V(\{u_k\}), V^*)$. From theorem 4 it follows that there exists K and D', such that $\{u_k(0)\}$ belongs to the trapping set $N = N(V_0, K, \gamma, D')$. Observe that $N \subset W(D', \gamma)$, hence we can pass to the limit with solutions obtained from Galerkin projections (see theorem 13). ∎

Proof of theorem 19: The proof is essentially the same as for theorem 18, the only difference is: we use theorem 5 instead of theorem 4. ∎

Proof of theorem 20: The global existence was proved in theorem 18. To prove the estimate for $|u_k(t_0)|$ we use theorem 6 to obtain

$$|u_k(t_0)| \leq \frac{D'}{|k|^\gamma} e^{-a|k|t_0}, \tag{77}$$

which finishes the proof. ∎

Theorem 21 *$d = 2$. If $u_0 \in C^5$ then the classical solution of NS equations such that $u(0, x) = u_0(x)$ exists for all $t > 0$ and it is analytic in space variables for $t > 0$.*

Proof: From lemma 14 it follows that the Fourier coefficients of u_0, $\{u_{0,k}\}$, satisfy assumptions of the theorem 18 with $\gamma = 5$. Hence there exists a solution, $\{u_k(t)\}$, of (8) in H, such that $u_k(0) = u_{0,k}$.

Let us set $u(t, x) = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} u_k(t) e^{ikx}$. It is easy to see that $u(t, x)$ is a classical solution of the Navier-Stokes system, because the Fourier series for all terms in the NS equations converge fast enough (compare proof of lemma 7).

From the theorem 20 and lemma 16 it follows that the function $u(t_0, \cdot)$ is analytic in space variables for any $t_0 > 0$. ∎

The following theorem is an easy consequence of theorem 13.

Theorem 22 *Assume $d = 2$ and $\gamma > 3$. Let W be any of the trapping regions defined in theorems 4 and 5, then the Navier-Stokes system induces a semidynamical system on W.*

7 Trapping regions in 3D

The goal of this section is to present method by Sannikov and Kaloshin [S] for constructing a trapping region for small initial data.

Let us state a result, which is not contained in [S], but can be easily obtained using the technique presented there.

We set the dimension $d = 3$. We assume the force f is zero.

Theorem 23 *For any $\gamma > 3.5$, there exists $D_0 = D_0(\gamma, \nu)$ such that for all $D < D_0$, if*

$$|u_k(0)| \leq \frac{D}{|k|^\gamma} \tag{78}$$

then the solution of (8) is defined for all $t > 0$ and

$$|u_k(t)| \leq \frac{D}{|k|^\gamma}, \quad t > 0 \tag{79}$$

Proof: Let

$$W = \left\{ \{u_k\} \mid |u_k| \leq \frac{D}{|k|^\gamma} \right\}. \tag{80}$$

From lemma 3 it follows that for $\{u_k\} \in W$ we have

$$\frac{d|u_k|}{dt} \leq \left| \sum (u_{k_1}|k) \sqcap_k u_{k-k_1} \right| - \nu |k|^2 |u_k| \leq \frac{D^2 C_Q(3,\gamma)}{|k|^{\gamma-1}} - \nu |k|^2 |u_k|. \tag{81}$$

Hence W is a trapping region if for every k we have

$$\frac{D^2 C_Q(3,\gamma)}{|k|^{\gamma-1}} - \frac{\nu D}{|k|^{\gamma-2}} < 0. \tag{82}$$

We obtain

$$\frac{D C_Q(3,\gamma)}{\nu} < |k|, \quad k \in \mathbb{Z}^3 \setminus \{0\}. \tag{83}$$

Hence if

$$D < D_0 = \frac{\nu}{C_Q(3,\gamma)}, \tag{84}$$

then W is a trapping region for all projections of the Navier Stokes equations. From lemma 17 it follows that the conditions C1,C2,C3 are satisfied (it is easy to see that condition D holds if $\gamma > 4$.) Hence we can pass to the limit with the dimension of Galerkin projection to obtain a desired solution. ∎

One can easily state similar theorem for analytic initial condition.

Let us comment on the Sannikov and Kaloshin result [S]. They constructed the trapping region of the form $|u_k| \leq \frac{D}{|}k|^2 e^{-v|k|t}$, $t \geq 0$, where $v > 0$. The methods developed in this paper require more compactness at $t = 0$ to be directly applicable to this trapping region.

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